

Generic Approximation of functions by their Padé approximants, I

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Abstract

Approximation of entire functions by their Padé approximants has been examined in the past. It is true that generically such an approximation holds. However, examining this problem from another viewpoint, we obtain stronger generic results on functions defined on simply connected domains or even open sets of arbitrary connectivity.

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1. Introduction

Every holomorphic function on a disc can be approximated by the partial sums of its Taylor development. Further, in any simply connected domain quasi all holomorphic functions are limits of a subsequence of their partial sums in the topology of uniform convergence on compacta ([6], [7]).

Instead of considering approximation by the partial sums of the Taylor developments, which are polynomials, one can examine the same question using rational functions, namely the Padé approximants $[p/q]_f$, $p, q \geq 0$ (see [1]). In [2] it is proved that quasi all entire functions $f \in H(\mathbb{C})$ are the limit of a subsequence $[p_n/q_n]_f$, where $p_n - q_n \rightarrow \infty$. Inspired by [3] we examine the same problem from a different scope and we obtain the same result under the weaker assumption $p_n \rightarrow +\infty$.

The same proof works if \mathbb{C} is replaced by a simply connected open set $\Omega \subset \mathbb{C}$ containing 0. This is done in Section 3 of the present article. We also mention that the

condition $p_n - q_n \rightarrow \infty$ (or $p_n \rightarrow +\infty, q_n \rightarrow +\infty$) is used in Theorem 5.1 of [3], and our result improves a corollary of Th. 5.1 of [3].

Finally, we mention that when we do approximation by polynomials, the maximum principle leads us to consider compact sets K with connected complement. However, when we do approximation by rational functions, as the Padé approximants, we may have poles on the holes of the compact set K . Thus, the result of Section 3 can be generalized to the case of open sets $\Omega \subset \mathbb{C}$ containing 0 of arbitrary connectivity, under the assumption $p_n \rightarrow +\infty, q_n \rightarrow +\infty$. This is done in Section 4. of the present article.

Our method of proof uses Baire's Category Theorem. We refer to [5] and [4] for the role of Baire's Theorem in Analysis.

2. Preliminaries

Let $\Omega \subseteq \mathbb{C}$ be an open set. Define $K_n = \{z \in \mathbb{C} : \text{dist}(z, \Omega^c) \geq 1/n, |z| \leq n\}$, $n \in \mathbb{N}$.

Remark 2.1. i) K_n is a compact subset of Ω and $K_n \subseteq K_{n+1}^0, \forall n$

ii) $\bigcup_{n=1}^{\infty} K_n = \Omega$ and if $K \subseteq \Omega$ compact $\exists n_0 \in \mathbb{N}: K_{n_0} \supseteq K$.

iii) Every component of $\mathbb{C} \cup \{\infty\} \setminus K_n$ contains at least one component of $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ ([9]).

We define a metric ρ on the set $H(\Omega)$ (of holomorphic in Ω):

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{\|f - g\|_{K_n}, 1\},$$

where $\|\cdot\|_{K_n}$ denotes the supremum norm on K_n . It is easy to see that a sequence in $H(\Omega)$, $(f_m)_{m \in \mathbb{N}}$ converges $f_m \xrightarrow{\rho} f$, if and only if $f_m \rightarrow f$ uniformly on the compact subsets of Ω . The space $(H(\Omega), \rho)$ is a complete metric space.

Let f be a function holomorphic in a neighborhood of 0 and let $f(z) = \sum_{v=0}^{\infty} a_v z^v$ its Taylor series. A Padé approximant $[p/q]_f$ of f , $p, q \in \{0, 1, 2, \dots\}$, is a rational function of the form

$$\frac{\sum_{v=0}^p n_v z^v}{\sum_{v=0}^q d_v z^v}, \quad d_0 = 1.$$

such that its Taylor series $\sum_{v=0}^{\infty} b_v z^v$ coincides with $\sum_{v=0}^{\infty} a_v z^v$ up to the first $p+q+1$ terms; that is $b_v = a_v$ for $v = 0, \dots, p+q$ ([1]).

In case of $q = 0$ there exists always a unique Padé approximant of f and $[p/q]_f(z) = S_p(z)$, where $S_p(z) = \sum_{v=0}^p a_v z^v$. For $q \geq 1$ it is true that there exists a unique Padé approximant of f , if and only if the following determinant is not zero:

$$\det \begin{vmatrix} a_{p-q+1} & a_{p-q+2} & \cdots & a_p \\ a_{p-q+2} & a_{p-q+3} & \cdots & a_{p+1} \\ \vdots & \vdots & & \vdots \\ a_p & a_{p+1} & \cdots & a_{p+q-1} \end{vmatrix} \neq 0, \quad a_i = 0, \quad \text{when } i < 0. \quad (*)$$

Then we write $f \in D_{p,q}$.

If $f \in D_{p,q}$, then $[p/q]_f$ ($q \geq 1$) is given by the Jacobi explicit formula:

$$[p/q]_f = \frac{\det \begin{vmatrix} z^q S_{p-q}(z) & z^{q-1} S_{p-q+1} & \cdots & S_p(z) \\ a_{p-q+1} & a_{p-q+2} & \cdots & a_{p+1} \\ \vdots & \vdots & & \vdots \\ a_p & a_{p+1} & \cdots & a_{p+q} \end{vmatrix}}{\det \begin{vmatrix} z^q & z^{q-1} & \cdots & 1 \\ a_{p-q+1} & a_{p-q+2} & \cdots & a_{p+1} \\ \vdots & \vdots & & \vdots \\ a_p & a_{p+1} & \cdots & a_{p+q} \end{vmatrix}},$$

$$\text{with } S_k(z) = \begin{cases} \sum_{v=0}^k a_v z^v, & k \geq 0 \\ 0, & k < 0. \end{cases}$$

If K is any set we write $f \in H(K)$ if f is holomorphic in some open set containing K .

Lemma 2.2. *Let $\Omega \subseteq \mathbb{C}$ be an open set, $0 \in \Omega$, and $\lambda \in \mathbb{N}$ such that $\overline{\Delta(0,r)} \subseteq K_\lambda^0$ for a certain $r > 0$. If $f \in H(K_\lambda)$, $f \in D_{p,q}$ and its Padé approximant has no poles in K_λ and if $\varepsilon > 0$ is given, then there exists $\delta > 0$ such that for every $g \in H(K_\lambda)$ with $\|g - f\|_{K_\lambda} < \delta$ it holds $g \in D_{p,q}$ and $\|[p/q]_g - [p/q]_f\|_{K_\lambda} < \varepsilon$.*

Proof. Observe that the above determinant $(*)$ and the coefficients of the numerator and the denominator of $[p/q]_f$ depend polynomially on $\frac{f^{(v)}(0)}{v!}$, $v = 0, 1, \dots, p+q$. This implies that there exists a $\tilde{\delta} > 0$ such that for every $g \in H(K_\lambda)$ with $|\frac{g^{(v)}(0)}{v!} - \frac{f^{(v)}(0)}{v!}| < \tilde{\delta}$, $v = 0, 1, \dots, p+q$ it holds $g \in D_{p,q}$ and $\|[p/q]_g - [p/q]_f\|_{K_\lambda} < \varepsilon$.

If $0 < \delta < \min\{r^v \cdot \tilde{\delta} \mid v = 0, 1, \dots, p+q\}$ and $\|g - f\|_{K_\lambda} < \delta$, then by Cauchy's estimates we obtain:

$$\left| \frac{g^{(v)}(0)}{v!} - \frac{f^{(v)}(0)}{v!} \right| = \left| \frac{(g-f)^{(v)}(0)}{v!} \right| \leq \frac{\|g-f\|_{\overline{\Delta(0,r)}}}{r^v} \leq \frac{\|g-f\|_{K_\lambda}}{r^v} < \frac{\delta}{r^v} < \tilde{\delta}. \quad \blacksquare$$

Remark 2.3. It follows that $D_{p,q}$ is open in $H(\Omega)$.

Remark 2.4. If all of the coefficients $\frac{f^{(v)}(0)}{v!} = a_v$, $v = 0, 1, \dots, p+q$, involved in the determinant $(*)$ depend linearly on $d \in \mathbb{C}$, $a_v = c_v \cdot d + \tau_v$, such that $c_v = 0$, when $v < p$ and $c_p \neq 0$, then the determinant $(*)$ is a polynomial in d of order q and hence only for finite values of d the determinant is zero.

3. The simply connected case

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain containing 0. Also, let $F \subseteq \mathbb{N} \times \mathbb{N}$ which contains a sequence $(\tilde{p}_m, \tilde{q}_m)_{m \in \mathbb{N}}$, such that $\tilde{p}_m \rightarrow +\infty$. We define

- $B_F = \{f \in H(\Omega) : \text{there exists } (p_m, q_m)_{m \in \mathbb{N}} \text{ in } F \text{ such that } f \in D_{p_m, q_m}, \text{ for all } m \in \mathbb{N} \text{ and for every } K \subseteq \Omega \text{ compact } [p_m/q_m]_f \rightarrow f \text{ uniformly on } K\}$.
- $E(n, s, (p, q)) = \{f \in H(\Omega) : f \in D_{p,q} \text{ and } \|[p/q]_f - f\|_{K_n} < 1/s\}$, $n, s \in \mathbb{N}$.

Lemma 3.1. $B_F = \bigcap_{n,s=1}^{\infty} \bigcup_{(p,q) \in F} E(n, s, (p, q))$.

Proof. It is standard and is omitted. [A similar proof can be found in [8]]. ■

Lemma 3.2. $E(n, s, (p, q))$ is open.

Proof. $D_{p,q}$ is open (Remark 2.3) and similarly as in Lemma 2.2, we can prove that the map $f \mapsto \|[p/q]_f - f\|_{K_n}$ is continuous, according to the Jacobi formula combined with Cauchy estimates. The lemma follows easily. ■

Theorem 3.3. B_F is G_δ and dense. (Hence $B_F \neq \emptyset$).

Proof. Lemma 3.2 implies that $\bigcup_{(p,q) \in F} E(n, s, (p, q))$ is open. By Lemma 3.1 B_F is G_δ . We claim that $\bigcup_{(p,q) \in F} E(n, s, (p, q))$ is dense. If that is true, then Baire's Category theorem completes the proof. By Runge's theorem the polynomials are dense in $H(\Omega)$. Therefore, it suffices to prove that for every polynomial P and $\varepsilon > 0$ there exists $f \in \bigcup_{(p,q) \in F} E(n, s, (p, q))$ such that $\|f - P\|_{K_N} < \varepsilon$, where $N = N(\varepsilon) \in \mathbb{N}$.

- Let P be a polynomial and $(p, q) \in F$ such that $p > \deg P$.

If $q = 0$, define $f(z) = P(z) + dz^p$, $d \in \mathbb{C} \setminus \{0\}$. It is immediate that $f \in D_{p,q}$ and $[p/q]_f = f$. It follows $f \in E(n, s, (p, q))$. Furthermore, $\|f - P\|_{K_N} = |d| \cdot \|z\|_{K_N}^p < \varepsilon$, when $0 < |d| < \varepsilon / \|z\|_{K_N}^p$.

If $q \geq 1$, we define $\tilde{f}(z) = \frac{P(z) + dz^p}{1 - (cz)^q}$, $d, c \in \mathbb{C} \setminus \{0\}$, where c and d will be determined later on.

- Let $\lambda \in \mathbb{N}$, $\lambda > \max\{n, N\}$, such that $K_\lambda^0 \supseteq \overline{\Delta(0, r)}$, where $r > 0$ is fixed.
- We have $\inf_{K_\lambda} |1 - (cz)^q| \geq 1 - |c|^q \cdot \|z\|_{K_\lambda}^q > \frac{1}{2}$, when $0 < |c| < \left(\frac{1}{2\|z\|_{K_\lambda}^q}\right)^{1/q}$.
- There exists $\tilde{\delta} > 0$ such that $\|\tilde{f} - P\|_{K_\lambda} = \sup_{z \in K_\lambda} \frac{|(cz)^q \cdot P(z) + d \cdot z^p|}{|1 - (cz)^q|} \leq 2(|c|^q \cdot \|z\|_{K_\lambda}^q \cdot \|P(z)\|_{K_\lambda} + |d| \cdot \|z\|_{K_\lambda}^p) < \varepsilon/2$, whenever $|d| < \tilde{\delta}$ and $|c| < \tilde{\delta} < \left(\frac{1}{2\|z\|_{K_\lambda}^q}\right)^{1/q}$.
- Around 0, $\tilde{f}(z) = P(z) + dz^p + P(z) \cdot (cz)^q + dz^p \cdot (cz)^q + \dots$. We fix a constant c satisfying the above. According to Remark 2.4 we can choose $0 < |d| < \tilde{\delta}$ such that $\tilde{f} \in D_{p,q}$. By the uniqueness of the Padé approximant of \tilde{f} we obtain $[p/q]_{\tilde{f}} = \tilde{f}$.
- Applying Lemma 2.2 there exists $0 < \delta < \min\{1/2s, \varepsilon/2\}$ such that, if $f \in H(K_\lambda)$ with $\|\tilde{f} - f\|_{K_\lambda} < \delta$ it follows $f \in D_{p,q}$ and $\|[p/q]_{\tilde{f}} - [p/q]_f\|_{K_\lambda} < 1/2s$. By Runge's theorem we can choose f to be a rational function with poles only in $(\mathbb{C} \cup \{\infty\}) \setminus K_\lambda$. More particularly Remark 2.1 enables us to choose f with pole only at ∞ because $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ is connected. Thus, f is a polynomial and $f \in H(\Omega)$. We also have
- $\|[p/q]_f - f\|_{K_n} \leq \|[p/q]_f - f\|_{K_\lambda} \leq \|[p/q]_f - [p/q]_{\tilde{f}}\|_{K_\lambda} + \|\tilde{f} - f\|_{K_\lambda} < 1/2s + \delta < 1/s$. It follows that $f \in E(n, s, (p, q))$ and $\|f - P\|_{K_N} \leq \|f - P\|_{K_\lambda} \leq \|f - \tilde{f}\|_{K_\lambda} + \|\tilde{f} - P\|_{K_\lambda} < \delta + \frac{\varepsilon}{2} < \varepsilon$.

This completes the proof. \blacksquare

Remark 3.4. In the above proof we have not used the fact that Ω is connected. Therefore Theorem 3.3 is valid for any simply connected open set $\Omega \subseteq \mathbb{C}$ containing 0.

4. Domains of arbitrary connectivity

Let $\Omega \subseteq \mathbb{C}$ be an open set containing 0. Also, let $F \subseteq \mathbb{N} \times \mathbb{N}$ containing a sequence $(\tilde{p}_m, \tilde{q}_m)_{m \in \mathbb{N}}$ such that $\tilde{p}_m \rightarrow +\infty$ and $\tilde{q}_m \rightarrow +\infty$. We define B_F and $E(n, s, (p, q))$ similarly as in Section 3.

The analogue of Lemmas 3.1, 3.2 hold in this case also and the proofs are similar.

Theorem 4.1. B_F is G_δ and dense. (Hence $B_F \neq \emptyset$).

Proof. Since $\bigcup_{(p,q) \in F} E(n, s, (p, q))$ is open, it follows that B_F is G_δ . In order to use Baire's Category theorem we will prove that $\bigcup_{(p,q) \in F} E(n, s, (p, q))$ is dense. By Runge's

theorem the rational functions with poles in $\Omega^c \cup \{\infty\}$ are dense in $H(\Omega)$. Therefore, it suffices to prove that for every rational function R with poles off Ω and $\varepsilon > 0$ there exists an $f \in \bigcup_{(p,q) \in F} E(n, s, (p, q))$ such that $\|f - R\|_{K_N} < \varepsilon$, where $N = N(\varepsilon) \in \mathbb{N}$.

- Let $R(z) = \frac{A(z)}{B(z)}$ be a rational function with poles only in $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ where A, B are polynomials. There is $(p, q) \in F$ such that $p > \deg A$ and $q > \deg B$. We define $\tilde{f}(z) = \frac{A(z) + dz^p}{B(z) - (cz)^q}$, $d, c \in \mathbb{C} \setminus \{0\}$ where c and d will be determined later on. Let $\lambda \in \mathbb{N}$, $\lambda > \max\{n, N\}$ such that $K_\lambda^0 \supseteq \overline{\Delta(0, r)}$ for some $r > 0$.
- Since R has no poles in Ω , it follows $B(0) \neq 0$ and $\inf_{z \in K_\lambda} |B(z)| > 0$.
- We have $\inf_{z \in K_\lambda} |B(z) - (cz)^q| \geq \inf_{z \in K_\lambda} |B(z)| - |c|^q \cdot \|z\|_{K_\lambda}^q > 0$, when $0 < |c| < \frac{(\inf_{z \in K_\lambda} |B(z)|)^{1/q}}{\|z\|_{K_\lambda}}$.
- There exists $\tilde{\delta} > 0$ such that $\|\tilde{f} - R\|_{K_\lambda} \leq \frac{\|A(z)\|_{K_\lambda} \cdot |c|^q \cdot \|z\|_{K_\lambda}^q + |d| \cdot \|z\|_{K_\lambda}^p \cdot \|B(z)\|_{K_\lambda}}{\inf_{z \in K_\lambda} |B(z)| \cdot \inf_{z \in K_\lambda} |B(z) - (cz)^q|} < \varepsilon/2$ whenever $|d| < \tilde{\delta}$ and $|c| < \tilde{\delta} < \frac{(\inf_{z \in K_\lambda} |B(z)|)^{1/q}}{\|z\|_{K_\lambda}}$.
- Around 0 we have: $\tilde{f}(z) = B^{-1}(0) \cdot A(z) + B^{-1}(0) \cdot dz^p - B^{-1}(0)A(z) \cdot (B^{-1}(0)\tilde{B}(z) - 1) - dz^p B^{-1}(0)(B^{-1}(0)\tilde{B}(z) - 1) + \dots$ where $\tilde{B}(z) = B(z) - (cz)^q$. We fix c satisfying the above. Then by Remark 2.4 we can choose $0 < |d| < \tilde{\delta}$ such that $\tilde{f} \in D_{p,q}$. Thus, there exists a unique Padé approximant of \tilde{f} and

$$\tilde{f}(z) = \frac{B^{-1}(0) \cdot A(z) - B^{-1}(0)dz^p}{B^{-1}(0) \cdot B(z) - B^{-1}(0)(cz)^q} \text{ satisfies } [p/q]_{\tilde{f}} = \tilde{f}.$$

- Lemma 2.2 provides $0 < \delta < \min\{1/2s, \varepsilon/2\}$ such that for every $f \in H(K_\lambda)$ with $\|\tilde{f} - f\|_{K_\lambda} < \delta$ it follows $f \in D_{p,q}$ and $\|[p/q]_{\tilde{f}} - [p/q]_f\|_{K_\lambda} < 1/2s$. By Runge's theorem there exists a rational function f with poles only in $(\mathbb{C} \cup \{\infty\}) \setminus K_\lambda$ which satisfies $\|\tilde{f} - f\|_{K_\lambda} < \delta$. The Remark 2.1 allows us to choose the poles in $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ and hence $f \in H(\Omega)$.
- We have $\|[p/q]_f - f\|_{K_n} \leq \|[p/q]_f - f\|_{K_\lambda} \leq \|[p/q]_f - [p/q]_{\tilde{f}}\|_{K_\lambda} + \|\tilde{f} - f\|_{K_\lambda} < 1/2s + \delta < 1/s$. It follows $f \in E(n, s, (p, q))$ and $\|f - R\|_{K_N} \leq \|f - R\|_{K_\lambda} \leq \|f - \tilde{f}\|_{K_\lambda} + \|\tilde{f} - R\|_{K_\lambda} < \delta + \varepsilon/2 < \varepsilon$. ■

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